Efficient Algorithms for Path Problems in Weighted Graphs

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Thesis Proposal

November 14, 2007

Thesis Committee:
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Weighted Graph Path Problems – Introduction
Our Problem: find a path $S = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k = T$ optimizing a given measure.
Path Measures

Shortest Paths: Find $S = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k = T$ minimizing

$$\sum_{i=0}^{k-1} w(v_i, v_{i+1}).$$
Path Measures

Shortest Paths: Find $S = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k = T$ minimizing

$$\sum_{i=0}^{k-1} w(v_i, v_{i+1}).$$

Application: find shortest road distance between two cities on a map.
Path Measures Cont.

Maximum Bottleneck Paths:

Find $S = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k = T$ maximizing

$$\min_{i=0}^{k-1} w(v_i, v_{i+1}).$$
Path Measures Cont.

Maximum Bottleneck Paths:
Find \( S = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k = T \) maximizing

\[
\min_{i=0}^{k-1} w(v_i, v_{i+1}).
\]

Application: find road path of highest tunnel clearance between two cities.
Minimum Nondecreasing Paths:

Find $S = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_{k-1} \rightarrow v_k = T$ such that $w(v_i, v_{i+1}) \leq w(v_{i+1}, v_{i+2})$ for all $i$, minimizing $w(v_{k-1}, T)$. 

![Diagram of a weighted graph with labeled edges and nodes. The red path indicates the minimum nondecreasing path from $S$ to $T$.](image)
Path Measures Cont.

Minimum Nondecreasing Paths:
Find $S = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_{k-1} \rightarrow v_k = T$ such that $w(v_i, v_{i+1}) \leq w(v_{i+1}, v_{i+2})$ for all $i$, minimizing $w(v_{k-1}, T)$.

Application: compute train itinerary which gets you from one city to another as early as possible.
Maximum Node Weighted Triangle:

In a node-weighted graph, if \((T, S')\) is an edge, find \(S \rightarrow v \rightarrow T\) maximizing

\[w(S) + w(v) + w(T).\]
Path Measures Cont.

Maximum Node Weighted Triangle:
In a node-weighted graph, if \((T, S)\) is an edge, find \(S \to v \to T\) maximizing \(w(S) + w(v) + w(T)\).

Application: Find important clusters in a database.
Problem Versions
Problem Versions

Single Source, Single Destination ($S$-$T$ Best Path)
Problem Versions

Single Source (every destination) Best Path

\[
\begin{array}{c}
S \\
\vdots \\
\end{array}
\]
Problem Versions

All Pairs Best Path

Diagram of a graph with weighted edges.
Talk Outline

1. Algorithms for All Pairs Best Paths
2. Algorithms for Single Source Best Paths
3. Directions for Further Research
All Pairs Path Problems – Results

\(n\)-number of vertices, \(m\)-number of edges

<table>
<thead>
<tr>
<th>Problem</th>
<th>Previous Best</th>
<th>Our Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>AP Max Triangle</td>
<td>(n^3)</td>
<td>(n^{2.58}) (VWY 2006)</td>
</tr>
<tr>
<td>AP Max Bottleneck Paths</td>
<td>(n^3)</td>
<td>(n^{2.79}) (VWY 2007)</td>
</tr>
<tr>
<td>AP Min Nondecreasing Paths</td>
<td>(n^3)</td>
<td>(n^{2.9}) (V 2008)</td>
</tr>
<tr>
<td>(k) Bits of Distance Product</td>
<td>(n^3 / \log n) (Chan 2005)</td>
<td>(2^k n^{2.69}) (VW 2006).</td>
</tr>
</tbody>
</table>
All Pairs Path Problems – Outline

1. Path Problems and Matrix Products
2. Properties and Algorithms
3. Techniques
4. Example
5. Summary of results
Path Problems and $\otimes$ Products

For all these problems, the length of $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ is

$$\ell(v_1, \ldots, v_k) = \ell(v_1, \ldots, v_{k-1}) \otimes w(v_{k-1}, v_k), \quad \ell(v_1, v_2) = w(v_1, v_2),$$

where $\otimes$ is different for each problem.

<table>
<thead>
<tr>
<th>Problem</th>
<th>$\otimes$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shortest Paths</td>
<td>$+$</td>
</tr>
<tr>
<td>Bottleneck Paths</td>
<td>min</td>
</tr>
<tr>
<td>Nondecreasing Paths</td>
<td>$\leq'$</td>
</tr>
</tbody>
</table>

$a \leq' b$ returns $b$ if $a \leq b$ and $\infty$ otherwise.
Path Problems and Matrices

\[
\begin{pmatrix}
\infty & 1 & 2 & \infty \\
\infty & \infty & \infty & 3 \\
\infty & \infty & \infty & 2 \\
\infty & \infty & \infty & \infty \\
\end{pmatrix}
\]
shortest paths

\[
\begin{pmatrix}
-\infty & 1 & 2 & -\infty \\
-\infty & -\infty & -\infty & 3 \\
-\infty & -\infty & -\infty & 2 \\
-\infty & -\infty & -\infty & -\infty \\
\end{pmatrix}
\]
bottleneck paths

\[
\begin{pmatrix}
\varepsilon_0 & 1 & 2 & \varepsilon_0 \\
\varepsilon_0 & \varepsilon_0 & \varepsilon_0 & 3 \\
\varepsilon_0 & \varepsilon_0 & \varepsilon_0 & 2 \\
\varepsilon_0 & \varepsilon_0 & \varepsilon_0 & \varepsilon_0 \\
\end{pmatrix}
\]
in general
Path Problems and Matrix Products

**Definition** \([\{\oplus, \otimes\}\text{-Product}]:\)

Given two \(n \times n\) real matrices \(A\) and \(B\), and two operations \(\otimes\) and \(\oplus\) on \(\mathbb{R}\), such that \(\oplus\) is commutative and associative, the \([\{\oplus, \otimes\}\text{-product} of A \text{ and } B\) is the \(n \times n\) matrix \(C\) given by

\[
C[i, j] := \bigoplus_{k=1}^{n} (A[i, k] \otimes B[k, j]), \forall i, j = 1, \ldots, n.
\]

For our path problems, \(\oplus\) is always \text{max} or \text{min}.
Various Matrix Products: definitions
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Algebraic Product:

\[ C[i, j] = (A \cdot B)[i, j] = \sum_k \{ A[i, k] \cdot B[k, j] \}. \]
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\[ C[i, j] = (A \star B)[i, j] = \min_k \{ A[i, k] + B[k, j] \}. \]
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\[ C[i, j] = (A \odot B)[i, j] = |\{ k : A[i, k] \leq B[k, j] \}|. \]
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Algebraic Product: \((\text{CoppersmithWinograd90})\)
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\[ C[i, j] = (A \odot B)[i, j] = | \{ k : A[i, k] \leq B[k, j] \} |. \quad n^{3 + \omega \over 2} \]

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Properties and Path Algorithms

Let $OPT[x, y]$ be the best $x \rightarrow y$ path length. $OPT[x, x] = \varepsilon_1$.

**Edge-Padding Property:**
There are operations $\oplus$ and $\otimes$, such that $\oplus$ is commutative and associative, and for all pairs of vertices $x, y$ in the graph,

$$OPT[x, y] = \bigoplus_{z \in V} (OPT[x, z] \otimes w(z, y)).$$
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This holds when $\otimes$ is right-distributive over $\oplus$. 
Midpoint Property:

If $x \rightarrow v_1 \rightarrow \ldots \rightarrow v_k \rightarrow y$ is an optimal path, then for all $i$,

$x \rightarrow \ldots \rightarrow v_i$ and $v_i \rightarrow \ldots \rightarrow y$ are also optimal.
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There are operations \( \oplus \) and \( \otimes \), where \( \oplus \) is associative and commutative, so that for all pairs of vertices \( x, y \) in the graph,

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Properties and Path Algorithms Cont.

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so that for all pairs of vertices \( x, y \) in the graph,
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\]

This holds when \( \otimes \) is associative and distributes over \( \oplus \).
Properties and Path Algorithms Cont.

When optimal paths have length at most $N$: 
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If the edge-padding property holds, all pairs best paths can be done in $O(N \times T[\langle \oplus, \otimes \rangle\text{-product}])$. 

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When $(\mathbb{R}, \oplus, \otimes, \varepsilon_0, \varepsilon_1)$ form a semiring, then all pairs best paths can be done in $O(T[({\oplus, \otimes})\text{-product}])$ (AHU74).
Matrix Products and Path Algorithms
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Still, we can show that if \((\min, \leq)\) product is in subcubic time, so is AP Minimum Nondecreasing Paths. \((\text{short paths - long paths method})\)
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Hence, we can concentrate on matrix products.
Techniques for Computing Matrix Products
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1. **Bucketting**: Preprocess each input matrix and assign its entries in a 1-to-1 fashion to some number of buckets, each getting a small number of entries. For each input matrix $X$ and each bucket $b$, create a new matrix $X_b$. 
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2. **Bucket Processing**: For each bucket $b$, multiply $A_b$ and $B_b$ using (perhaps recursively) a different matrix product ($\oplus'$, $\otimes'$).
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2. **Bucket Processing:** For each bucket $b$, multiply $A_b$ and $B_b$ using (perhaps recursively) a different matrix product $(\oplus', \otimes')$.

3. **Exhaustive Search:** The bucket processing step provides information which allows us to choose a small number of buckets on which the problem is solved by exhaustive search.
Example: \((\min, \leq)\)-Product

We want \(a_{ij} = \min_k \{B[k, j] \mid A[i, k] \leq B[k, j]\}\).

We will use the dominance product: \((A \odot B)[i, j] = |\{k : A[i, k] \leq B[k, j]\}|\).
**Example: \((\min, \leq)\)-Product**

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1. Take the columns of \( B \) and sort the entries of each column.

\[
B = \begin{pmatrix}
10 & 2 & 0 & 7 \\
-1.1 & 3 & -1 & 2.1 \\
5.1 & 7 & -2 & 4 \\
3.2 & 1 & -3 & 2.1 \\
\end{pmatrix}
\]

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1. Take the columns of \(B\) and sort the entries of each column.

2. (Technique 1) Bucket the entries of each column of \(B\), in their sorted order into \(s\) roughly equal buckets.

\[
B = \begin{pmatrix}
10 & 2 & 0 & 7 \\
-1.1 & 3 & -1 & 2.1 \\
5.1 & 7 & -2 & 4 \\
3.2 & 1 & -3 & 2.1
\end{pmatrix}
\]

column 1: \(A[2, 1], A[4, 1], A[3, 1], A[1, 1]\)

column 2: \(A[4, 2], A[1, 2], A[2, 2], A[3, 2]\)

column 3: \(A[5, 3], A[3, 3], A[2, 3], A[1, 3]\)

column 4: \(A[4, 4], A[2, 4], A[3, 4], A[1, 4]\)
Example: \((\min, \leq)\)-Product cont.

3. (Technique 1 - **Bucketting**) For each bucket \(b\) create a matrix \(B(b)\) containing only the elements in bucket \(b\) and \(-\infty\) in all other entries.

\[
B(1) = \begin{pmatrix}
-\infty & 2 & -\infty & -\infty \\
-1.1 & -\infty & -\infty & 2.1 \\
-\infty & -\infty & -2 & -\infty \\
3.2 & 1 & -3 & 2.1
\end{pmatrix}
\quad \quad
B(2) = \begin{pmatrix}
10 & -\infty & 0 & 7 \\
-\infty & 3 & -1 & -\infty \\
5.1 & 7 & -\infty & 4 \\
-\infty & -\infty & -\infty & -\infty
\end{pmatrix}
\]
Example: \((\min, \leq)\)-Product cont.

We want \(a_{i,j} = \min_k \{B[k, j] \mid A[i, k] \leq B[k, j]\}\).

We will use the dominance product: \((A \odot B)[i, j] = |\{k : A[i, k] \leq B[k, j]\}|\).

4. (Technique 2 - **Bucket Processing**) Compute \(A \odot B(b)\) for each \(b\).

\[
B \odot B(2) =
\begin{pmatrix}
10 & 2 & 0 & 7 \\
-1.1 & 3 & -1 & 2.1 \\
5.1 & 7 & -2 & 4 \\
3.2 & 1 & -3 & 2.1
\end{pmatrix}
\odot
\begin{pmatrix}
10 & -\infty & 0 & 7 \\
-\infty & 3 & -1 & -\infty \\
5.1 & 7 & -\infty & 4 \\
-\infty & -\infty & -\infty & -\infty
\end{pmatrix}
= 
\begin{pmatrix}
2 & 2 & 0 & 1 \\
2 & 2 & 1 & 2 \\
2 & 1 & 0 & 2 \\
2 & 2 & 0 & 2
\end{pmatrix}
\]

This tells us for every bucket \(b\) and each \(i, j\), the number of coords \(k\) such that \(B[k, j]\) is in bucket \(b\) and \(A[i, k] \leq B[k, j]\).

This step takes \(O(sn^{3+\omega \over 2})\) since dominance product takes \(O(n^{3+\omega \over 2})\).
Example: \((\min, \leq)\)-Product cont.
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5. For each \(i, j\) we know the smallest bucket \(b\) in which there is an entry 
\(B[k, j]\) such that \(A[i, k] \leq B[k, j]\).
Example: \((\min, \leq)\)-Product cont.

5. For each \(i, j\) we know the smallest bucket \(b\) in which there is an entry \(B[k, j]\) such that \(A[i, k] \leq B[k, j]\).

6. (Technique 3 - Exhaustive Search) For each \(i, j\), search that bucket for smallest \(B[k, j]\) - there are at most \(O(n/s)\) entries we have to go through for each pair \(i, j\).

This step takes \(O(n^3/s)\) and explicitly finds witnesses.
Example: \((\min, \leq)\)-Product cont.

5. For each \(i, j\) we know the smallest bucket \(b\) in which there is an entry \(B[k, j]\) such that \(A[i, k] \leq B[k, j]\).

6. (Technique 3 - Exhaustive Search) For each \(i, j\), search that bucket for smallest \(B[k, j]\) - there are at most \(O(n/s)\) entries we have to go through for each pair \(i, j\).

   This step takes \(O(n^3/s)\) and explicitly finds witnesses.

7. The overall runtime is minimized for \(s = n^{3-\omega/4}\) and the runtime is then \(O(n^{9+\omega/4}) = O(n^{2.85})\).
Summary of Results
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All Pairs Maximum Node Weighted Triangles - either by using dominance product, or by using rectangular (algebraic) matrix multiplication. Best Running Time: $O(n^{2.575})$. (VW STOC06, VWY ICALP06)
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All Pairs Shortest Paths - compute $k$ most significant bits of the distance product in $O(2^k n^{2.688})$ time using dominance product. (VW STOC06)
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All Pairs Minimum Nondecreasing Paths - compute $(\min, \leq)$-Product using dominance product, and then apply short path - long path technique of Zwick/Chan. Best Running Time: $O(n^{2.896})$. (V SODA08)
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All Pairs Bottleneck Paths - compute $(\max, \min)$-Product using $(\min, \leq)$-Product. Best Running Time: $O(n^{2.792})$. (VWY STOC07)
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Single Source Bottleneck Paths - A $O(m + n)$ algorithm is known for the undirected single source, single target version (Punnen91). In the general case: Dijkstra’s with Fibonacci Heaps $O(m + n \log n)$. 
Directions for Further Research

1. More single source algorithms - better single source algorithm for bottleneck paths?

2. Parallel Algorithms

3. Combinatorial Algorithms
Parallel Algorithms

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Hence our all pairs algorithms running sequentially in $O(n^c)$ time can be done in parallel on $O(n^c)$ processors and $O(\text{poly log } n)$ time.
Parallel Algorithms Cont.

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Conjecture: for $K$-source bottleneck / nondecreasing paths reduce preprocessing work to $O((n + s(n)^\alpha) \log n)$–work where:

$\alpha = 2.792$ for bottleneck and $\alpha = 2.896$ for nondecreasing paths.
Purely Combinatorial Algorithms

Purely combinatorial – nonalgebraic, nonsubtractive.
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Examples: Four Russians Algorithm for Matrix Multiplication in $O(n^3 / \log^2 n)$, Chan algorithm for all pairs shortest paths in $O(n^3 \log \log^3 n / \log^2 n)$. 
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**Examples:** Four Russians Algorithm for Matrix Multiplication in $O(n^3 / \log^2 n)$, Chan algorithm for all pairs shortest paths in $O(n^3 \log \log^3 n / \log^2 n)$.

We want similar runtimes for AP bottleneck paths, and AP nondecreasing paths.
Purely Combinatorial Algorithms Cont.
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The somewhat sparse case – $O(mn \log(n^2/m)/\log^2 n)$ algorithm for matrix multiplication, transitive closure and max weight triangle (BVW 08).

Previous best was $O(mn/\log n)$ (Chan06). Ours is better when $m = n^{2-o(1)}$. 
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Similar algorithms for AP shortest, AP nondecreasing, or AP bottleneck paths?
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- **April** - thesis [defense?](#)
Thank you!