Hybrid Algorithms for Graph Problems

Joint work with Ryan Williams and Maverick Woo

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Defying Hardness using Graph Minors and Separators

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Introduction

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Approximation Ratio and Time, etc.

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h_2 solves the problem exactly but runs in subexponential time ($2^{o(n)}$).

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A *selector* S which on each instance selects in polynomial time the *best* heuristic.

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Defying Hardness: **Some NP-Hard problems are known or conjectured to be *hard* on several complexity measures m_i separately.**

E.g. Clique cannot be approximated within a factor of n^{ϵ}, and cannot be solved in polynomial time, unless P$=$NP.

There exist hybrid algorithms for NP-Hard problems which for each h_i (on the instances on which S chooses h_i to be run) do *strictly* better than the corresponding known hardness guarantees m_i.

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No better than $1/2$-approximation is known without using SDP.

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for each edge in M, with probability $1/2$ choose which of its endpoints to put in A. Put the other endpoint in B;

for each vertex v not covered by M, with probability $1/2$ choose whether to place it in A or B.

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If $|M| < \varepsilon \frac{m}{2}$,

M has at most εm vertices, and the rest of the vertices form an independent set I. Placing the vertices of I so that the cut is maximized, given an arrangement of M is easy.

We get an exact solution in $\tilde{O}(2^{\varepsilon m})$ time.

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The probability that an edge not in M crosses the cut is $1/2$. Hence we get a cut of expected size at least $(\varepsilon \frac{m}{2}) + \frac{1}{2}(m - \varepsilon \frac{m}{2}) = (\frac{1}{2} + \frac{\varepsilon}{4})m$.

We get a $(\frac{1}{2} + \frac{\varepsilon}{4})$-approximation with no semidefinite programming.

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We give a *hybrid* algorithm which for any $\ell(n)$

- either finds a path of length ℓ, or
- solves the Longest Path exactly in time$2^{O(\ell \log L \log \frac{n}{\ell})}$.

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Notice that for $\ell = n/polylog(n)$ we get subexponential exact running time and a polylog approximation.

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2. If P has length at least ℓ, stop and output P.
3. Else, remove f from P and add it to A.

   If $|A| = n/2$, stop and output P as a separator.

   Otherwise, attempt to continue P with vertices from $V - P - A$ until f with no neighbors is reached. Go to 2.

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The width of a tree decomposition is the maximum size of a bag W_i, minus 1. The *tree width* of a graph G is the minimum width of a tree decomposition of G.

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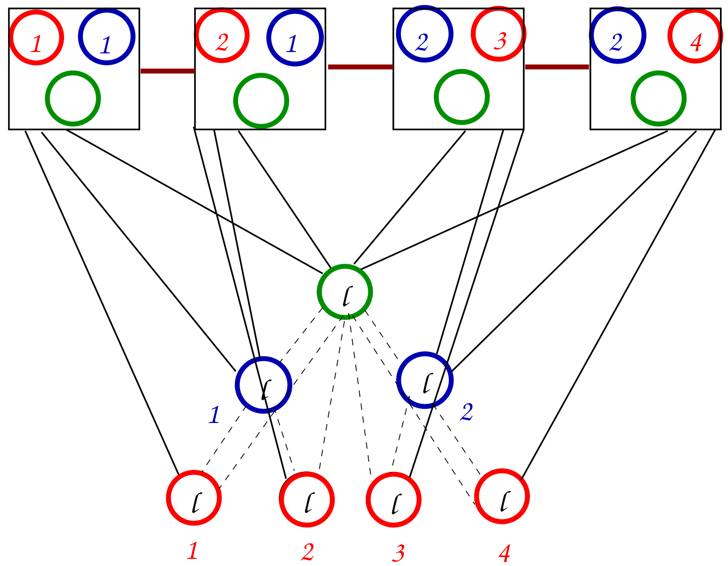

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4. Recurse on G_L and G_R to obtain either a path of length ℓ, or a separator decomposition.

5. Run the Matousek and Thomas algorithm on the tree decomposition obtained from the separator tree, on successive powers of 2 for the path length, to obtain the longest path in $\tilde{O}(2^{\ell \log L \log \frac{n}{\ell}})$.

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Often one says S is a $1/3 - 2/3$ – separator, meaning that in the worst case $|A| = \frac{1}{3}|V(G)|$ and $|B| = \frac{2}{3}|V(G)|$.

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For large values of ℓ the above can be generalized to finding a minor, or finding a $1/2 - 1/2$–separator.

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B is a subset of $V(G)$, $B \cap M = \emptyset$, and $|N(B) \cap W| \leq \frac{|B|}{\ell}$.

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5. Otherwise, the *smaller* of $R' = R \cup N(R)$, and $R'' = V - R'$ has *few neighbors in* W ($|N(R')| \leq \frac{|R'|}{\ell}$ and $|N(R'')| \leq \frac{|R''|}{\ell}$).
   We add it to B.

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The separator consists of the (unfinished) minor M and of the neighbors of B in W. Since $|N_W(B)| \leq |B|/\ell \leq \frac{2n}{3\ell}$, the size of the separator is $O(n/\ell + \ell h \log n)$.

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For large values of ℓ the above can be generalized to finding a minor, or finding a $1/2 - 1/2$-separator.

As in the case of Path-Separator we can obtain a separator tree, or an H-minor.

Minimum Bandwidth

Problem: Given a graph G, give a permutation π on the vertices of G so that the maximum edge *stretch* $\max_{(i,j)\in E(G)} |\pi(i) - \pi(j)|$ is minimized.

Best approximation: $O(\log^3 n\sqrt{\log\log n})$ by Dunagan and Vempala, 2001, $O(\sqrt{\frac{n}{B}}\log n)$ by Avrim Blum et al. where B is the optimum bandwidth

Best Exact Algorithm: $\tilde{O}(10^n)$ by Feige and Killian, 2000

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Gabber and Galil show how to construct 5–regular $(\frac{2-\sqrt{3}}{4})$–expanders efficiently.

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Let h_{LHS} and h_{RHS} be the *number* of supernodes completely contained among the first $k/2$ nodes (respectively, last $k/2$ nodes) in π.

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Let $h_S = h - h_{LHS} - h_{RHS}$.

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Each supernode is disjoint from other supernodes and is connected, so the arrangement has $\varepsilon \cdot h$ nodes in the first half that connect to distinct nodes in the second half.

Any arrangement with this property has bandwidth at least $\varepsilon \cdot h$.

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- $h_S \geq \varepsilon h/6$, which by the above implies the bandwidth is at least $\varepsilon \cdot h/6$, or
- there are at least $\varepsilon h/6$ first half neighbors in the second half, in which case there are $\varepsilon h/6$ edges crossing from nodes in the first half to *distinct* nodes in the second half, so again the bandwidth is at least $\Omega(h)$.

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  This guarantees that the bandwidth of G is large, and hence the $O(\sqrt{\frac{n}{B}} \log n)$–approximation algorithm by Avrim et al. gives a good approximation.

- Otherwise use the separator tree to get a good exact algorithm for bandwidth.

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- a $\log n$ bit index for the position of each separator node in the current allowed set of indices,
- a length n bit string specifying whether left or right subtree nodes go at the corresponding position. We recurse on the left and right subtree separately, using the positions specified by the corresponding bits.

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$T(n) = \tilde{O}(4^n \cdot n^{\ell \log(n/\ell)})$, and if ℓ is chosen to be small, say $o(\frac{n}{(\log n \log \log n)})$, $T(n) = 4^{n+o(n)}$.

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We gave a hybrid algorithm for Minimum Bandwidth which either approximates within $\alpha(n) \log^{2.5} n \log \log n$ (for unbounded $\alpha(n)$) or solves exactly in $4^{n+o(n)}$ time. This also beats the best known conventional algorithms on both accounts.

Thank You!