A Dominance Approach to Weighted Graph Problems

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Theory Lunch

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Introduction
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Naiive algorithm: $O(n^3)$, matrix mult.: $O(n^\omega) = O(n^{2.38})$. 

\[
G^3 = \begin{pmatrix}
2 & \cdot & \cdot & \cdot \\
\cdot & 2 & \cdot & \cdot \\
\cdot & \cdot & 2 & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{pmatrix}
\]
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Other examples: LP, exact algorithms for NP-hard problems, graph perfect matching, unweighted APSP.
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In general it is not clear how to speed-up weighted versions of problems in a similar way.

Example open problems include: maximum weighted matching, finding minimum weighted triangles and other patterns, weighted APSP.
Our approach \cite{VW06}

Instead of matrix multiplication we use the so called dominance product to speed-up weighted problems.

We demonstrate the approach on finding minimum weighted triangles, computing bits of the distance product, all pairs bottleneck paths.
Talk outline

1. Some definitions
2. Dominance product in subcubic time
3. Maximum weighted triangle
4. Computing bits of the distance product
5. All pairs bottleneck paths
6. Open problems
Various Matrix Products: definitions
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Algebraic Product:

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Dominance Product:
\[ C[i, j] = (A \odot B)[i, j] = |\{k : A[i, k] \leq B[k, j]\}|. \]
\[ = \sum_k (A[i, k] \leq B[k, j]). \]
How to compute the dominance product

Recall \((A \odot B)[i, j] = |\{k : A[i, k] \leq B[k, j]\}|\).
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**Thm.** (Matousek) Dominance Product can be computed in \(n^{(3+\omega)/2}\) time.

We sketch the elegant algorithm in the next few slides.

It uses fast matrix multiplication.
Dominance Product in $n^{(3+\omega)/2}$

$$(C[i, j] = |\{ k : A[i, k] \leq B[k, j]\}|)$$
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**Idea 1:** Just care about the sorted order of coordinates

$\implies$ WLOG each column of $A$ and the corresponding row of $B$ is a permutation of $[2n]$. 
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Make $n$ sorted lists $L_1, \ldots, L_n$, where

$L_k$ has the $k$th column of $A$ and the $k$th row of $B$
**Dominance Product in** $n^{(3+\omega)/2}$

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Partition each $L_k$ into “buckets” with $s$ elements in each bucket
Dominance Product in $n^{(3+\omega)/2}$, Cont.

\[
(C[i, j] = |\{k : A[i, k] \leq B[k, j]\}|
\]

Idea 2: Two types of data are counted in $C'$:

Dominance Product in $n^{(3+\omega)/2}$, Cont.

$$(C[i, j] = |\{k : A[i, k] \leq B[k, j]\}|)$$

Idea 2: Two types of data are counted in $C$:

1. Pairs $(A[i, k], B[k, j])$ such that $A[i, k] \leq B[k, j]$, but $A[i, k]$ and $B[k, j]$ fall in the same bucket of $L_k$
   - Only $O(n^2 s)$ possible pairs of this form
   - Can compute these in $O(1)$ amortized time
Dominance Product in $n^{(3+\omega)/2}$, Cont.

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**Idea 2:** Two types of data are counted in $C$:

1. Pairs $(A[i, k], B[k, j])$ such that $A[i, k] \leq B[k, j]$, but $A[i, k]$ and $B[k, j]$ fall in different buckets of $L_k$
   
   - Can count these using $2n/s$ matrix multiplications
     (One matrix multiply for each bucket)
Dominance computation step 2

For every $t = 1, \ldots, 2n/s$, create matrices $A_t$ and $B_t$ such that

$$A_t[i, k] = \begin{cases} 1 & \text{if } A[i, k] \text{ in bucket } t \text{ of } L_k \\ 0 & \text{otherwise} \end{cases}$$

$$B_t[k, j] = \begin{cases} 1 & \text{if } B[k, j] \text{ in bucket } s > t \text{ of } L_k \\ 0 & \text{otherwise} \end{cases}$$
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$$\sum_t A_t B_t$$

gives the pairs $A[i, k], B[k, j]$ such that $A[i, k] \leq B[k, j]$ and they are in different buckets of $L_k$.

This can be done in $n/s \cdot n^\omega$ time.
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This can be done in $n/s \cdot n^\omega$ time.

**Overall Runtime:** Pick $s : n^2 s = n/s \cdot n^\omega \iff s = n^{\omega-1/2}$.

The final running time is $O(n^{3+\omega/2}) = O(n^{2.69})$. 
Maximum node weighted triangle

**Input:** Graph with real-number weights on the nodes

**Task:** Find a triangle of maximum weight sum
Maximum node weighted triangle

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Task: Find a triangle of maximum weight sum

(Reduce Node-Weighted Triangle to Edge-Weighted Triangle):

Push weights from nodes to edges: \( w(u, v) = (w(u) + w(v))/2 \)
Folklore Result
Recall the **distance product** of $A$ and $B$ is

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Observation: **Distance Product can solve Max Weighted Triangle**
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**Observation:** Distance Product can solve Max Weighted Triangle

$\rightarrow$ Compute $MAX_{i,j}\{((-A) \star (-A))[i, j] - A[i, j]\}$

(Min Weight Triangle: $MIN_{i,j}\{(A \star A)[i, j] + A[i, j]\}$)
”Easy” Weighted Triangle Algorithms
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$\longrightarrow$ Max Weight Triangle in $O(M \cdot n^\omega)$ (Pseudopolynomial)
”Easy” Weighted Triangle Algorithms

- [Zwick, '02] $O(M \cdot n^\omega)$ distance product algorithm, $M$ is the largest weight of an edge
  $\implies$ Max Weight Triangle in $O(M \cdot n^\omega)$ (Pseudopolynomial)

- [Chan, '05] $O(n^3 / \log n)$ distance product
  $\implies$ Max Weighted Triangle in $O(n^3 / \log n)$
"Easy" Weighted Triangle Algorithms

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- [Chan, '05] $O(n^3 / \log n)$ distance product
  $\implies$ Max Weighted Triangle in $O(n^3 / \log n)$

Truly Sub-Cubic Algorithm?
Using the dominance product we get:

- Deterministic Algorithm [VW06]
  
  \[ O(B \cdot n^{(3+\omega)/2}) \leq O(B \cdot n^{2.688}) \text{, where } B \text{ is the bit precision} \]

- Randomized (Strongly Polynomial) Algorithm [VW06]
  
  \[ O(n^{(3+\omega)/2 \log n}) \leq O(n^{2.688}) \]
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**Aside:** It is already known how to find a max node weighted triangle in \( O(n^\omega) \) [CzumajLingas07].

We can get for all edges the max node weighted triangle including the edge in \( O(n^{2.58}) \) time [VWY06].
Deterministic Algorithm: Outline
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1. Does there exist a triangle of weight sum \textit{at least} $K$?
   \rightarrow \text{dominance product instance.}
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   → dominance product instance.

2. Do binary search on $K$ to find the maximum weight $W$ of a triangle.

3. Find a triangle of weight $W$. 
Step 1: Given $K$, reduce to dominance product instance.

Vertex $i \in V \rightarrow$
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Vertex $i \in V \rightarrow$

- row vector $A[i, ;] = (A[i, 1], \ldots, A[i, n])$ s.t.

$$A[i, j] = \begin{cases} K - w(i) & \text{if there is an edge from } i \text{ to } j \\ \infty & \text{otherwise} \end{cases}$$
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- column vector $B[; , i] = (B[1, i], \ldots, B[n, i])$ s.t.

$$B[j, i] = \begin{cases} w(i) + w(j) & \text{if there is an edge from } i \text{ to } j \\ -\infty & \text{otherwise.} \end{cases}$$
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$$A[i, j] \leq B[j, k] \iff K \leq w(i) + w(k) + w(j) \text{ and } (i, j), (j, k) \in E$$
Step 1 cont.
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\((A \otimes B)[i, k] \neq 0\) iff

\(\exists j\) such that there is a path \(i \rightarrow j \rightarrow k\) and \(w(i) + w(k) + w(j) \geq K\)
Step 1 cont.

\[(A \odot B)[i, k] \neq 0 \text{ iff } \exists j \text{ such that there is a path } i \rightarrow j \rightarrow k \text{ and } w(i) + w(k) + w(j) \geq K\]

Hence to check whether there is a triangle of weight at least \(K\), compute \(C = A \odot B\) and check for an entry \(C[i, j] \neq 0\) such that \((i, j) \in E\).
Runtime
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But this algorithm is not strongly polynomial because of the binary search.
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Then the binary search calls at most $O(B)$ dominance computations, and hence the runtime is $O(B \cdot n^{\frac{3+\omega}{2}})$.

But this algorithm is not strongly polynomial because of the binary search.

Can use random sampling of weighted triangles to obtain a $O(n^{\frac{3+\omega}{2}} \log n)$ strongly polynomial randomized algorithm.
The distance product

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The best algorithms for arbitrary real weights are

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- by Han in \(O(n^3 (\log \log n / \log n)^{5/4})\).
Computing bits of the distance product
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Suppose only need $B$ bits of $(A \star B)[i, j] = \min_k \{ A[i, k] + B[k, j] \}$. 
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For constant $K$, we can set up a matrix $A(K)$ s.t. for all $i, j$,

$$A(K)[i, j] = K - A[i, j].$$
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Compute $D(K) = (A(K) \odot B)$

and $C(K)[i, j] = \begin{cases} 1 & \text{if } D(K)[i, j] = n \\ 0 & \text{otherwise.} \end{cases}$
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$D(K)[i, j] \neq n \iff \exists k. K - A[i, k] > B[k, j]$.

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Computing bits of the distance product

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$\rightarrow D(K)[i, j] \neq n \iff \exists k. K - A[i, k] > B[k, j]$ and $C(K)[i, j] = \begin{cases} 1 & \text{if } D(K)[i, j] = n \\ 0 & \text{otherwise.} \end{cases}$

Then $C(K)[i, j] = 1 \iff \min_k (A[i, k] + B[k, j]) \geq K$.

Most significant bit is then $C\left(\frac{W}{2}\right)$ where $W$ is the smallest power of 2 larger than the largest distance.
Computing bits of the distance product
Computing bits of the distance product

\[ C(K)[i, j] = 1 \iff \min_k (A[i, k] + B[k, j]) \geq K \]

The second most significant bit of \((A \star B)[i, j]\) is

\[
(\neg C(W)[i, j] \land C\left(\frac{3W}{4}\right)[i, j]) \lor (\neg C\left(\frac{W}{2}\right)[i, j] \land C\left(\frac{W}{4}\right)[i, j]).
\]

Only compute 4 dominance products.
Computing bits of the distance product

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The second most significant bit of \((A \star B)[i, j]\) is

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Only compute 4 dominance products.

The \(\ell\)th bit is

\[ 2^{\ell-1} - 1 \bigvee_{s=0}^{2^{\ell-1} - 1} \left[ \neg C(W(1 - \frac{s}{2^{\ell-1}}))[i, j] \land C(W(1 - \frac{s}{2^{\ell-1}} - \frac{1}{2^\ell}))[i, j] \right]. \]

Here need \(O(2^\ell)\) dominance products.
Computing bits of the distance product

**Thm.** The first \( B \) most significant bits of the distance product of two \( n \times n \) matrices can be computed in \( O(2^B n^{3+\omega}) \) time.

One can compute \( (\frac{3-\omega}{2} - \varepsilon) \log n \) bits in \( O(n^{3-\varepsilon}) \) time.
Bottleneck paths

The bottleneck edge of a path in a graph from vertex $u$ to vertex $v$ is the edge of smallest weight.

In many applications (e.g. max flow), the path of maximum bottleneck is needed.

In this talk we will consider the all pairs max bottlenecks problem.
Bottleneck paths – related work

single source:

• Folklore: in $O(m + n \log n)$ by Dijkstra.

all pairs:

• Folklore: undirected edge weighted in $O(n^2)$ using min spanning tree.
• Shapira, Yuster, Zwick 2007: directed node weighted in $O(n^{2.58})$.
• VW: directed edge weighted in $O(n^{2.79})$. 
MaxMin product

Recall \((A \bullet B)[i, j] = \max_k \min\{A[i, k], B[k, j]\}\).
MaxMin product

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The **MaxMin product** is used to compute all pairs maximum bottleneck paths (**APBP**), similar to how one uses **distance product** for **APSP**.
MaxMin product

Recall $(A \bullet B)[i, j] = \max_k \min\{A[i, k], B[k, j]\}$.

The MaxMin product is used to compute all pairs maximum bottleneck paths (APBP), similar to how one uses distance product for APSP.

Computing the MaxMin product of two $n \times n$ matrices takes the same time as computing all pairs bottleneck distances in an $n$ vertex graph.
Computing the MaxMin product faster

\[ C = (A \bullet B)[i, j] = \max_k \min\{A[i, k], B[k, j]\} \]

We use the **dominance product** again:

\[ (A \odot B)[i, j] = |\{k : A[i, k] \leq B[k, j]\}|. \]

We will proceed as follows:
Computing the MaxMin product faster

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We will proceed as follows:

1. compute for all \(i, j\), \(a_{ij} = \max_k \{A[i, k] : A[i, k] \leq B[k, j]\}\),

2. compute for all \(i, j\), \(b_{ij} = \max_k \{B[k, j] : B[k, j] \leq A[i, k]\}\),
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We will proceed as follows:

1. compute for all \(i, j\), \(a_{ij} = \max_k \{A[i, k] \mid A[i, k] \leq B[k, j]\}\),
2. compute for all \(i, j\), \(b_{ij} = \max_k \{B[k, j] \mid B[k, j] \leq A[i, k]\}\),
3. set for all \(i, j\), \(C[i, j] = \max\{a_{ij}, b_{ij}\}\).
Computing the MaxMin product faster

We want \( a_{ij} = \max_k \{ A[i, k] \mid A[i, k] \leq B[k, j] \} \).

1. Take the rows of \( A \) and sort the entries of each row.

2. **Bucket** the entries of each row of \( A \), in their sorted order into \( s \) roughly equal buckets.

\[
A = \begin{pmatrix}
10 & -1.1 & 5.1 & 3.2 \\
2 & 3 & 7 & 1 \\
0 & -1 & -2 & -3 \\
7 & 2.1 & 4 & 2.1
\end{pmatrix}
\]

row 1 : \( A[1, 2], \ A[1, 4], \ A[1, 3], \ A[1, 1] \)

row 2 : \( A[2, 4], \ A[2, 1], \ A[2, 2], \ A[2, 3] \)

row 3 : \( A[3, 4], \ A[3, 3], \ A[3, 2], \ A[3, 1] \)

row 4 : \( A[4, 4], \ A[4, 2], \ A[4, 3], \ A[4, 1] \)
Computing the MaxMin product faster

3. For each bucket \( b \) create a matrix \( A(b) \) containing only the elements in bucket \( b \) and \( \infty \) in all other entries.

\[
A(1) = \begin{pmatrix}
\infty & -1.1 & \infty & 3.2 \\
2 & \infty & \infty & 1 \\
\infty & \infty & -2 & -3 \\
\infty & 2.1 & \infty & 2.1
\end{pmatrix}
\quad A(2) = \begin{pmatrix}
10 & \infty & 5.1 & \infty \\
\infty & 3 & 7 & \infty \\
0 & -1 & \infty & \infty \\
7 & \infty & 4 & \infty
\end{pmatrix}
\]
Computing the MaxMin product faster

4. Compute $A(b) \odot B$ for each bucket $b$.

$$A(2) \odot A = \begin{pmatrix} 10 & \infty & 5.1 & \infty \\ \infty & 3 & 7 & \infty \\ 0 & -1 & \infty & \infty \\ 7 & \infty & 4 & \infty \end{pmatrix} \odot \begin{pmatrix} 10 & -1.1 & 5.1 & 3.2 \\ 2 & 3 & 7 & 1 \\ 0 & -1 & -2 & -3 \\ 7 & 2.1 & 4 & 2.1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 2 & 2 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

This tells us for every bucket $b$ and each $i, j$, the number of coords $k$ such that $A[i, k]$ is in bucket $b$ and $A[i, k] \leq B[k, j]$.

This step takes $O(sn^{\frac{3+\omega}{2}})$. 
Computing the MaxMin product faster
Computing the MaxMin product faster

5. For each $i, j$ we know the largest bucket $b$ in which there is an entry $A[i, k]$ such that $A[i, k] \leq B[k, j]$. 
Computing the MaxMin product faster

5. For each $i, j$ we know the largest bucket $b$ in which there is an entry $A[i, k]$ such that $A[i, k] \leq B[k, j]$.

For each $i, j$, search that bucket for $k$ - there are at most $O(n/s)$ entries we have to go through for each pair $i, j$.

This step takes $O(n^3/s)$ and explicitly finds witnesses.
Computing the MaxMin product faster

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For each $i, j$, search that bucket for $k$ - there are at most $\mathcal{O}(n/s)$ entries we have to go through for each pair $i, j$.

This step takes $\mathcal{O}(n^3/s)$ and explicitly finds witnesses.

6. The overall runtime is maximized for $s = n^{3-\frac{\omega}{4}}$ and the runtime is then $\mathcal{O}(n^{\frac{9+\omega}{4}}) = \mathcal{O}(n^{2.81})$. 
Computing the MaxMin product faster

5. For each $i, j$ we know the largest bucket $b$ in which there is an entry $A[i, k]$ such that $A[i, k] \leq B[k, j]$.

For each $i, j$, search that bucket for $k$ - there are at most $O(n/s)$ entries we have to go through for each pair $i, j$.

This step takes $O(n^3/s)$ and explicitly finds witnesses.

6. The overall runtime is maximized for $s = n^{3-\omega}/4$ and the runtime is then $O(n^{9+\omega}/4) = O(n^{2.81})$.

7. You can do slightly better by using sparse dominance $\rightarrow O(n^{2.79})$. 
Open Problems

1. dominance product in $n^\omega$? (VW Conjecture)

2. truly subcubic distance product using dominance product?

3. generalize the technique for some class of problems?
Thank You!