Graduate Algorithms Midterm 2

Extra Credit

Virginia Vassilevska

April 2, 2004

Claim: Unless P=NP, one cannot count the number of simple cycles in a directed graph G in polynomial time.

Proof: Call the problem of counting simple cycles in a directed graph **#SimpleCycles**. I'll construct a reduction from Hamiltonian Cycle as follows:

Given an instance $G = (V, E)$ of (directed) Hamiltonian Cycle, replace each edge by the widget shown in Figure. The point of the widget is to let each edge (u, v) in G be replaced by 2^k distinct simple paths from u to v in the new graph G'. We'll see that $k = ((n-1)\log n + 1)$ is sufficient ($|V| = n,\ |E| = m$).

Now, suppose that G has a Hamiltonian Cycle, i.e. a cycle of length n. Then this cycle in G' is represented by $(2^k)^n$ distinct simple directed cycles. Hence the number of directed simple cycles in G' is at least 2^{kn}.

If, on the other hand, G has no Hamiltonian Cycle, then the largest a simple cycle can be in G is $(n-1)$. Hence, the number of simple cycles in G is at most n^{n-1}, and each cycle is represented by at most $(2^k)^{n-1}$ distinct simple cycles in G'. Also note that that all cycles in G' are some representation of a cycle of G.

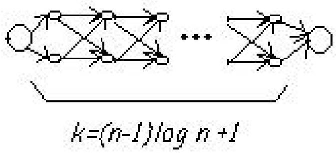

Figure 1: a widget to represent each edge

This is since the widget is acyclic. Hence the number of simple cycles in G' is at most $n^{n-1}(2^k)^{n-1}$.
Now note that
$n^{n-1}(2^k)^{n-1} = (2^{k+\log n})^{n-1} = 2^{kn}2^{((n-1)\log n - k)} = 2^{kn}2^{-1} < 2^{kn}$.

Also note that we added to each edge $2k$ vertices and $4k-1$ edges. For our choice of k this is clearly polynomial. Since the overall number of new vertices and edges we have to create are respectively $2km$ and $m(4k-1)$, and since it is easy to go through the edges and to change them to widgets the reduction is clearly polynomial.

Hence if we can solve the #SimpleCycles problem in polynomial time, then we can decide the Hamiltonian Cycle problem in polynomial time: for each G we count the cycles in G' and if there are less than 2^{kn}, then we answer no, else we answer yes. Since Hamiltonian Cycle is NP-complete, unless P=NP, #SimpleCycles has no polynomial time (deterministic or randomizing) algorithm.

Corollary 1 The problem of counting the number of simple directed cycles going through a particular vertex v of a graph G is not solvable in polynomial time unless P=NP. Suppose we have a polynomial time algorithm $A(v, G)$ to count the number of simple cycles through v in a graph G. Then let G be an instance of #SimpleCycles. Consider the following algorithm

- Set $cycles = 0$
- while $V(G)$ is nonempty, pick a vertex v
  1. $cycles = cycles + A(v, G)$
  2. remove v from G

I claim that the above correctly counts the number of simple cycles in G: each cycle is counted exactly once. Each cycle is counted when the first vertex it goes through is considered, and is counted never again. (if a cycle does not go through v, then removing v from the graph does not remove the cycle). The algorithm above goes through the loop exactly n times and if $A(v, G)$ takes polynomial time, then so will this algorithm. Hence, unless P=NP, we cannot count the number of simple cycles going through a particular vertex.

Corollary 2 The problem of counting the number of simple paths between two vertices s and t in a digraph G is not solvable in polynomial time unless P=NP.

To see this, take an instance of the problem of counting the number of simple cycles going through v in a digraph G and split v into two vertices v' and v'' so that v' has exactly the same out-neighbors as v (and no in-neighbors), and v'' has the same in-neighbors as v (but no out-neighbors) to create graph G'. The number of cycles going through v in G is the number of simple paths starting from v and ending at some in-neighbor of v, hence they are exactly the simple paths from v' to v'' in G'. Hence solving the path problem solves the fixed vertex cycle problem. And so the problem of counting the number of simple paths between two vertices s and t in a digraph G is not solvable in polynomial time unless P=NP.