A solution

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Problem: Starting from 0 we want to find a colored integer n on the real line by a continuous path, so that we minimize the ratio $(pathlength)/|n|$.

Claim: The best ratio one can get (in the limit) is 9, and this ratio is achieved by the algorithm that goes from 0 to $(-1)^i 2^i$ and back to 0 at steps $i = 0 \ldots \infty$.

Proof: First note, any algorithm that will find the solution must be able to cover the entire line. Since we start from 0, this means that the path one takes will have to change direction. Notice that if a change of direction of an optimal algorithm appears on one side of 0, then the next change of direction needs to take place on the other side. Otherwise we are covering an interval twice and we can shortcut the steps as shown in Fig. 1 to obtain a path which takes a smaller number of steps and covers the same points. Hence, the algorithm alternates between the two sides of 0. Let d_k be the distance of the kth switch in direction from 0. The d_k define the alternating algorithm.

Consider d_i and d_{i+1}. Suppose $d_{i+1} < d_i$, in particular, let for $k \geq 1$, $k < d_i$ $d_{i+1} = d_i - k$. Consider the algorithm A for which everything is the same as in OPT except that $d'_i = d_i - 1$. All of the ratios are at least as good as in OPT except the ratio for the point p at distance d_i from the origin (at the appropriate side). Let $S = \sum_{j=1}^{i-1} 2d_j$. Then the ratio for p and A is $1 + (S + 2(d_i - 1) + 2(d_i - k))/d_i$. Now, on the other side of the origin, the point at distance $d_{i+1} + 1$ has ratio for OPT at least $1 + (S + 2d_i + 2(d_i - k) + 2(d_i + 1))/(d_i - k + 1)$. This is $1 + (S + 6d_i - 2(k-1))/(d_i - k + 1)$.

We have $-(k-1)S \leq 0 \leq 4kd_i + 2 = 2d_i^2 + 4kd_i - 2d_i^2 + 2 \leq 2d_i^2 + 4kd_i - 2k^2 + 2 = 2d_i^2 + 4kd_i - 2(k-1)(k+1)$. So $Sd_i + 6d_i^2 - 2d_i(k-1) \geq Sd_i + 4d_i^2 - 2d_i(k+1) - 4d_i(k-1) + 2(k-1)(k+1) - (k-1)S$, and $(S + 6d_i - 2(k-1))/(d_i - k + 1) \geq (S + 4d_i - 2(k+1))/d_i$. This means that A is no worse than OPT.

Hence, we can assume that for all i, $d_{i+1} \geq d_i$. Now suppose we have $d_i = d_{i+1}$ for some i. Then, again, we can reduce d_i to $d_i - 1$ creating algorithm A. This is since the only ratio that will change is that for point p at distance d_i from the origin (at the respective side). The ratio will become $1 + (S + 4d_i - 2)/d_i$. For OPT the point at distance $d_{i+1} + 1$ has ratio at least $1 + (S + 6d_i + 2)/(d_i + 1)$. Now, $S \leq 4\sum_{j=1}^{d_i - 1} j = 2(d_i - 1)d_i < 2d_i^2 + 2$. Therefore, $1 + (S + 4d_i - 2)/d_i \leq 1 + (S + 6d_i + 2)/(d_i + 1)$, and A is at least as good as OPT.

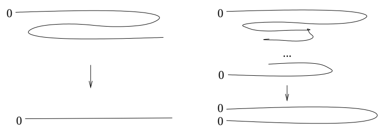

Figure 1:

Now we can assume that for all i, $d_{i+1} > d_i$.

From now on we consider the ratio $C = \max_i \sum_{j=1}^{i} d_j/(d_{i-1}+1)$ as representing the algorithm, and we want to minimize that. We will show that this ratio is at least 4, which gives $\max_i 1 + \frac{2\sum_{j=1}^{i} d_j}{(d_{i-1}+1)} \geq 9$ for the *pathlength*$/|n|$ ratio.

Now, we have $\max_i \sum_{j=1}^{i} d_j/(d_{i-1}+1) > (d_i + d_{i-1}+1)/(d_{i-1}+1) \geq (2d_{i-1}+2)/(d_{i-1}+1) = 2$. Hence $C > 2$. For an exponential function $d_i' = k^i$ we have that $\frac{k^{i-1}-1}{(k-1)(k^{i-2}+1)} < \frac{k^i-1}{(k-1)(k^{i-1}+1)}$, and so the ratio grows for larger i. Hence the max ratio is $\lim_{i\to\infty} \frac{k^i-1}{(k-1)(k^{i-1}+1)} = \frac{k^2}{k-1}$. Suppose $\frac{k^2}{k-1} = C$. Then $k = \frac{C\pm\sqrt{C^2-4C}}{2}$. Notice that k is real only when $C \geq 4$. Hence the smallest ratio representable by exponential functions is 4 (and this is achieved by 2^i, in the limit). We need to show that the ratio that any other function represents is at least 4.

We also note that $\max_i \sum_{j=1}^{i} d_j/(d_{i-1}+1) = 1 + \max_i \frac{d_i+\sum_{j=1}^{i-2} d_j}{d_{i-1}+1}$. From now on we'll look at $\max_i \frac{d_i+\sum_{j=1}^{i-2} d_j}{d_{i-1}+1}$ which we'll show cannot be smaller than 3.

We consider $r_i = \frac{d_i+\sum_{j=1}^{i-2} d_j}{d_{i-1}+1}$. Suppose $r_i < 3$ for all i. This is the same as $\frac{d_i-3+\sum_{j=1}^{i-2} d_j}{d_{i-1}} < 3$. This means $\frac{d_i+\sum_{j=3}^{i-2} d_j}{d_{i-1}} < 3$ for all i. Now let $z_i = \frac{d_{i-1}}{\sum_{j=3}^{i-2} d_j}$. Then $\frac{d_i+\sum_{j=3}^{i-2} d_j}{d_{i-1}} = \frac{1}{z_i} + \frac{d_i}{d_{i-1}} = \frac{1}{z_i} + (\frac{d_i}{\sum_{j=3}^{i-1} d_j})(\frac{\sum_{j=3}^{i-1} d_j}{d_{i-1}}) = \frac{1}{z_i} + z_{i+1}(1+\frac{1}{z_i}) < 3$. Hence $z_{i+1} < \frac{3z_i-1}{z_i+1}$. Let $g(z) = \frac{3z-1}{z+1}$ and $g^i(z)$ be g composed with itself i times. For any $z \neq 1$ $G_z(i) = g^i(z)$ is strictly decreasing and if $z < 1$ it goes below 1/2. This means that if at some i $z_i < 1$, then in the limit, $z_\infty \leq 1/2$ and hence $r_\infty \geq 3$. Furthermore, if for some i we have $z_i = 1$ then $z_{i+1} < 1$ and we still get $r_\infty \geq 3$. Now, we also know that $z_i < 3$, as if $z_i = \frac{d_{i-1}}{\sum_{j=3}^{i-2} d_j} \geq 3$, then $\frac{d_{i-1}}{d_{i-2}} \geq 3$ and hence $r_{i-1} \geq 3$. We have $z_{i+2k-1} < \frac{k+1}{k}$. This means $z_\infty = 1$, or we've reached 1/2 at some i (or at ∞). Now, $z_\infty = 1$ means $r_\infty = \lim_{i\to\infty} \frac{d_i+\sum_{j=3}^{i-2} d_j}{d_{i-1}} = 1 + \lim_{i\to\infty} \frac{2\sum_{j=3}^{i-2} d_j}{d_{i-1}} = 3$.