All Pairs Bottleneck Paths in Truly Subcubic Time

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joint work with Ryan Williams and Raphael Yuster
Introduction
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This talk: truly subcubic algorithm for APBP – studied alongside APSP.
Bottleneck paths - definitions

Given: graph $G = (V, E)$ with arbitrary edge weights $w : E \to \mathbb{R}$.

The bottleneck edge of a path in $G$ from vertex $u$ to vertex $v$ is the edge of smallest weight on the path.
Maximum bottleneck paths

In many applications (e.g. max flow), the path of maximum bottleneck is needed.

In this talk we will consider the all pairs max bottlenecks problem: for all pairs of vertices $s$ and $t$ in the graph, find the weight of the maximum bottleneck edge on a path from $s$ to $t$. 
All pairs bottleneck paths – related work

• Pollack 1960: introduced APBP and showed a cubic algorithm.

• Hu 1961: undirected, edge weighted – max spanning tree. $O(n^2)$

• Shapira, Yuster, Zwick 2007: directed, node weighted in $O(n^{2.58})$.

• this work: directed, edge weighted in $O(n^{2.79})$. 
MaxMin product

The MaxMin product of two $n \times n$ matrices $A$ and $B$ is

$$(A \bullet B)[i, j] = \max_k \min_{k} \{A[i, k], B[k, j]\}.$$
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Moreover: computing the \textbf{MaxMin product} of two $n \times n$ matrices takes the same time (asymptotically) as computing \textbf{all pairs bottleneck weights} in an $n$ vertex graph. [AhoHopcroftUllman74]
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Moreover: computing the **MaxMin product** of two $n \times n$ matrices takes the same time (asymptotically) as computing all pairs bottleneck weights in an $n$ vertex graph. [AhoHopcroftUllman74]

This work: first truly subcubic algorithm for the MaxMin product.
MaxMin product in subcubic time

MaxMin: \( C = (A \bullet B)[i, j] = \max_k \min\{A[i, k], B[k, j]\} \)

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1. compute for all \( i, j \), \( a_{ij} = \max_k \{A[i, k] \mid A[i, k] \leq B[k, j]\} \),
2. compute for all \( i, j \), \( b_{ij} = \max_k \{B[k, j] \mid B[k, j] \leq A[i, k]\} \),
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3. set for all \( i, j \), \( C[i, j] = \max\{a_{ij}, b_{ij}\} \).
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$$(A \odot B)[i, j] = |\{k : A[i, k] \leq B[k, j]\}|.$$

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Thm. (Matousek) Dominance Product can be computed in \( O(n^{(3+\omega)/2}) \) time, where \( \omega \) is the exponent of fast matrix multiplication. \( \leftarrow O(n^{2.69}) \)
MaxMin product in subcubic time

We want \( a_{ij} = \max_k \{ A[i, k] \mid A[i, k] \leq B[k, j] \} \).

1. Take the rows of \( A \) and sort the entries of each row.

2. Bucket the entries of each row of \( A \), in their sorted order into \( s \) roughly equal buckets.

\[
A = \begin{pmatrix}
10 & -1.1 & 5.1 & 3.2 \\
2 & 3 & 7 & 1 \\
0 & -1 & -2 & -3 \\
7 & 2.1 & 4 & 2.1
\end{pmatrix}
\]

row 1: \( A[1, 2], A[1, 4], A[1, 3], A[1, 1] \)
MaxMin product in subcubic time

3. For each bucket $b$ create a matrix $A(b)$ containing only the elements in bucket $b$ and $\infty$ in all other entries.

$$A(1) = \begin{pmatrix}
\infty & -1.1 & \infty & 3.2 \\
2 & \infty & \infty & 1 \\
\infty & \infty & -2 & -3 \\
\infty & 2.1 & \infty & 2.1 \\
\end{pmatrix} \quad A(2) = \begin{pmatrix}
10 & \infty & 5.1 & \infty \\
\infty & 3 & 7 & \infty \\
0 & -1 & \infty & \infty \\
7 & \infty & 4 & \infty \\
\end{pmatrix}$$
MaxMin product in subcubic time

Recall, \((A \otimes B)[i, j] = |\{k : A[i, k] \leq B[k, j]\}|\). 

4. Compute \(A(b) \otimes B\) for each bucket \(b\).

\[
A(2) \otimes A = \begin{pmatrix}
10 & \infty & 5.1 & \infty \\
\infty & 3 & 7 & \infty \\
0 & -1 & \infty & \infty \\
7 & \infty & 4 & \infty \\
\end{pmatrix} \otimes \begin{pmatrix}
10 & -1.1 & 5.1 & 3.2 \\
2 & 3 & 7 & 1 \\
0 & -1 & -2 & -3 \\
7 & 2.1 & 4 & 2.1 \\
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
2 & 1 & 2 & 2 \\
1 & 0 & 0 & 0 \\
\end{pmatrix}
\]

This tells us for every bucket \(b\) and each \(i, j\), the number of coords \(k\) such that \(A[i, k]\) is in bucket \(b\) and \(A[i, k] \leq B[k, j]\).

This step takes \(O(sn^{3+\omega/2})\).
MaxMin product in subcubic time
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5. For each $i, j$ we find the largest bucket $b$ in which there is an entry $A[i, k]$ such that $A[i, k] \leq B[k, j]$. 
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For each $i, j$, search that bucket for $k$ - there are at most $O(n/s)$ entries we have to go through for each pair $i, j$.

This step takes $O(n^3/s)$ and explicitly finds witnesses.
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6. The overall runtime is maximized for $s = n^{\frac{3-\omega}{4}}$ and the runtime is then $O(n^{\frac{9+\omega}{4}}) = O(n^{2.85})$. 
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7. You can do slightly better by using sparse dominance $\rightarrow O(n^{2.79})$. 
Sparse dominance

**Theorem:** Let $A$ and $B$ be $n \times n$ matrices with entries from a totally ordered set. Let $S \subseteq [n] \times [n]$ such that $|S| = m \geq n$. Let $C$ be the matrix such that

$$C[i, j] = \left| \{k \mid (i, k) \in S \text{ and } A[i, k] \leq B[k, j] \} \right|.$$

There is an algorithm that, given $A$, $B$, and $S$, outputs $C$ in $O(\sqrt{m} \cdot n^{\frac{1+\omega}{2}})$ time.

**Intuition:** The set $S$ of coordinate pairs contains all entries of $A$ we care about. Comparisons between entries of $A$ not in $S$ and entries of $B$ are ignored.
MaxMin Product

Recall the matrices $A(b)$:

$$A(1) = \begin{pmatrix}
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$A(b)$ has $O(n^2/s)$ finite entries. Each of the $s$ dominance products thus takes $O(n^{3+\omega}/\sqrt{s})$, and the running time for the entire algorithm is: $O(n^3/s + \sqrt{sn^{3+\omega}})$, minimized for $s = n^{1-\omega/3}$. 
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In $O(n^{2+\frac{\omega}{3} \log n})$ time one can obtain a witness matrix from which one can obtain **actual paths** in time linear in their length.
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**Open Problems**

1. dominance product, MaxMin product in $O(n^{\omega})$?

2. truly subcubic distance product using dominance product?
Thank You!